A Calculus of Variations Approach to Determining the Functions that Describe the Curves of Hanging Chains and Springs.

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Calculus of variations originated with Johann Bernoulli proposing the Brachistochrone problem in 1696. The solution to this problem would be the path, in the form of a function, that minimizes the sliding time of a particle on a frictionless path between two points. There were solutions proposed by many of the famous mathematicians of the day (Euler, Newton, Leipniz, and Jakob Bernoulli… to name a few). The end result is a method of minimizing a functional that appears throughout classical mechanics. We applied this method to hanging systems.

To determine the function that describes a hanging system it is necessary to write an integral that describes the systems potential energy. Because this system will be hanging such that the potential energy of the system is at minimum, calculus of variations can be used to find the function for possible curves.

This project consists of two parts. The first half of the experiment is solving the hanging chain system parameterized with respect to angle of inclination (after first being parameterized with respect to arc length along the core curve). The second half dealt with solving the hanging spring system parameterized with respect to arc length along the wire.

Part I: The Hanging Chain

The energy in a hanging chain is determined by gravitational potential energy and an energy related to the interaction of the links (a form of bending energy). Since there are many small links the contribution from this energy is small. Because this energy is very difficult to model, it has been left out and some error can be attribute to its absence.

We parameterized the energy equation with respect to arc length because we felt it would provide us with valuable experience such that we would then be better able to deal with the necessary parameterization of the spring system. The resulting energy equation is a function of a constant density, gravity, and the vertical component of the parameterized system. We then further parameterized the system with respect to angle of inclination because it would simplify the differential equation in the Euler-Lagrange equation.

$$\dot{\gamma} = (\cos[\theta],\ \sin[\theta])$$

$$x(s) = \int_0^s \cos[\theta[\sigma]]\, d\sigma$$

$$y(s) = \int_0^s \sin[\theta[\sigma]]\, d\sigma$$

This is the parameterization that was used.

It was then necessary to define the constraints on the system before the final energy equation could be written. The system is set up such that the starting point for each curve is at the origin. The constraints are therefore the maximum values for the vertical and horizontal parameters.

$$\int_0^L \cos[\theta[\sigma]]\, d\sigma = b$$

$$\int_0^L \sin[\theta[\sigma]]\, d\sigma = h$$

The constraints applied to the energy equation.

The energy equation was then complete and the first variation could be applied to it.

$$E(\theta, \sigma) = \int_0^L y[s]\, ds = \int_0^L \int_0^s \sin[\theta[\sigma]]\, d\sigma\, ds = \int_0^L (L-\sigma) \sin[\theta[\sigma]]\, d\sigma$$

The energy equation and its parameterization.

$$E(\theta, \sigma) = \int_0^L ((L-\sigma) \sin[\theta[\sigma]] + \lambda\cos[\theta[\sigma]] + \mu\sin[\theta[\sigma]])\, d\sigma$$

$$= \int_0^L ((L-\sigma+\mu) \sin[\theta[\sigma]] + \lambda\cos[\theta[\sigma]])\, d\sigma$$

The energy equation with constraint applied to it.

The energy equation with the variation applied to it yielded a trigonometric Euler-Lagrange equation (this simple solution arose from the use of the angle of inclination to describe the system).

$$(L-\sigma+\mu)\cos[\theta[\sigma]] = \lambda\sin[\theta[\sigma]]$$

$$\tan[\theta[\sigma]] = \frac{L+\mu-\sigma}{\lambda}$$

$$\theta[\sigma] = \tan^{-1}\left[\frac{L+\mu-\sigma}{\lambda}\right]$$

The Euler-Lagrange equation and its solution.

Before going any further, since we knew that the solution to this problem must be in the form of a hyperbolic cosine curve, it was then possible to test if this result was reasonable by working backwards from parameterizing a cosh function.

$$\dot{\gamma} = (\cos[\theta], \sin[\theta])$$

$$x(s) = \int_0^s \cos[\theta[\sigma]]\, d\sigma$$

$$y(s) = \int_0^s \sin[\theta[\sigma]]\, d\sigma$$

The parameterization used throughout Part 1 of the project.

$$y = \cosh[x]$$

The solution of the hanging chain problem is some modified version of this expression, so parameterizing this equation should yield a function of arctan for the angle of inclination.

$$\int_0^s \sin[\theta[\sigma]]\, d\sigma = \cosh\left[\int_0^s \cos[\theta[\sigma]]\, d\sigma\right]$$

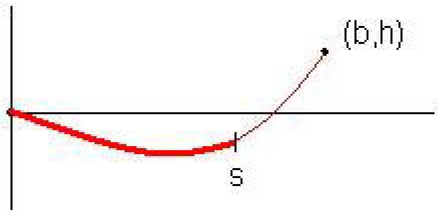
The equation after parameterization.

$$\begin{aligned}
&\arctan\left(\frac{\left(e^{\mathrm{RootOf}\left((e^{_Z})^4 + 2(e^{_Z})^2 + 1 - 4s^2(e^{_Z})^2 + 4_Z^2(e^{_Z})^2\right)}\right)^2 + 1}{s\, e^{\mathrm{RootOf}\left((e^{_Z})^4 + 2(e^{_Z})^2 + 1 - 4s^2(e^{_Z})^2 + 4_Z^2(e^{_Z})^2\right)}}, \mathrm{RootOf}\Big(\right. \\
&\left(e^{\mathrm{RootOf}\left((e^{_Z})^4 + 2(e^{_Z})^2 + 1 - 4s^2(e^{_Z})^2 + 4_Z^2(e^{_Z})^2\right)}\right)^4 \\
&+ 2\left(e^{\mathrm{RootOf}\left((e^{_Z})^4 + 2(e^{_Z})^2 + 1 - 4s^2(e^{_Z})^2 + 4_Z^2(e^{_Z})^2\right)}\right)^2 + 1 \\
&- 4s^2\left(e^{\mathrm{RootOf}\left((e^{_Z})^4 + 2(e^{_Z})^2 + 1 - 4s^2(e^{_Z})^2 + 4_Z^2(e^{_Z})^2\right)}\right)^2 + _Z^2, \mathit{label} = _L3\Big) \Big/ \Big(\\
&\left. e^{\mathrm{RootOf}\left((e^{_Z})^4 + 2(e^{_Z})^2 + 1 - 4s^2(e^{_Z})^2 + 4_Z^2(e^{_Z})^2\right)} s\Big) \right)
\end{aligned}$$

The preceding is the solution Maple returns when the parameterized equation is solved for θ. This is a function of arctan, which, with the right parameters, would simplify to our result. To prove that answer is in the form of an arctan function is enough to establish that our previous result is logical.

From the solution to the Euler-Lagrange Equation, we derived two transcendental equations based on our constraints that would yield the Lagrange multiplier coefficients for the constraints. Once the constraints are applied to the transcendental equations and Lagrange multipliers are known, it is possible to solve for the function, y(x), that describes the curve of the hanging chain.

An Example:



$$b = \int_0^L \sin\left[\tan^{-1}\left[(L+\mu-\sigma)/(\lambda)\right]\right] d\sigma$$

$$= \frac{\sqrt{\lambda^2+\mu^2}\sqrt{1+\frac{(L+\mu)^2}{\lambda^2}}\,\mathrm{Log}\left[-\mu+\sqrt{\lambda^2+\mu^2}\right]-\sqrt{L^2+\lambda^2+2L\mu+\mu^2}\sqrt{1+\frac{\mu^2}{\lambda^2}}\,\mathrm{Log}\left[-L-\mu+\sqrt{L^2+\lambda^2+2L\mu+\mu^2}\right]}{\sqrt{1+\frac{\mu^2}{\lambda^2}}\sqrt{1+\frac{(L+\mu)^2}{\lambda^2}}}$$

$$h = \int_0^L \cos\left[\tan^{-1}\left[(L+\mu-\sigma)/(\lambda)\right]\right] d\sigma = \frac{\frac{-\lambda^2-\mu^2}{\sqrt{1+\frac{\mu^2}{\lambda^2}}}-\frac{-L^2-\lambda^2-2L\mu-\mu^2}{\sqrt{1+\frac{(L+\mu)^2}{\lambda^2}}}}{\lambda}$$

The transcendental equations.

$$b = \frac{-\sqrt{\lambda^2+\mu^2}\sqrt{1+\frac{(61.4`+\mu)^2}{\lambda^2}}\,\mathrm{Log}\left[0.`+\mu+\sqrt{\lambda^2+\mu^2}\right]+\sqrt{1+\frac{\mu^2}{\lambda^2}}\sqrt{\lambda^2+(61.4`+\mu)^2}\,\mathrm{Log}\left[61.4`+\mu+\sqrt{\lambda^2+(61.4`+\mu)^2}\right]}{\sqrt{1+\frac{\mu^2}{\lambda^2}}\sqrt{1+\frac{(61.4`+\mu)^2}{\lambda^2}}}$$

$$h = \frac{\frac{0.`-\lambda^2+0.`\mu-\mu^2}{\sqrt{1+\frac{(0.`+\mu)^2}{\lambda^2}}}-\frac{-3769.96`-\lambda^2-122.8`\mu-\mu^2}{\sqrt{1+\frac{(61.4`+\mu)^2}{\lambda^2}}}}{\lambda}$$

The transcendental equations with b=45.0 cm, h = 6.0 cm, and L=61.4 cm

$$\lambda = -16.11843349193664`$$

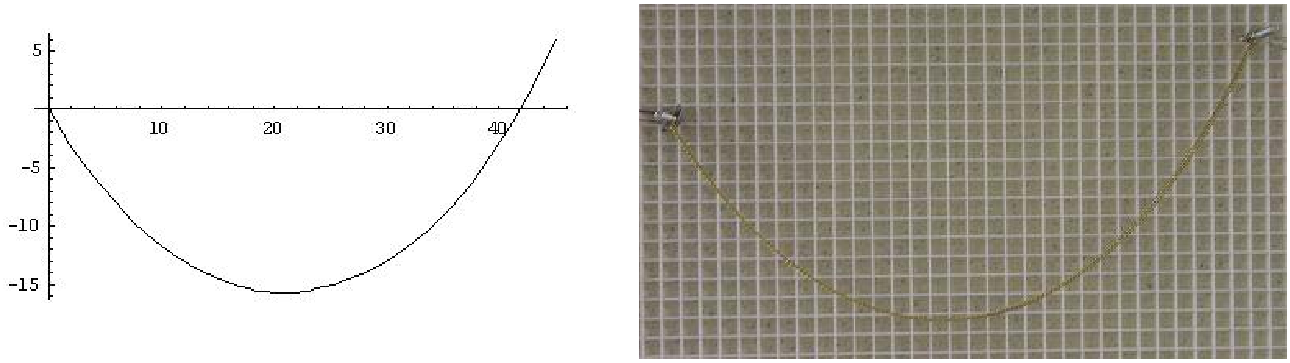
$$\mu = -34.09187727106603`$$

The solution to the transcendental equations.

$$x = \int_0^s \cos[\arctan[(61.4-34.09187727106603`-\sigma)/(-16.11843349193664`)]]\, d\sigma$$

$$s = 6.34009*10^{-16}\left(4.30721*10^{16}-2.5423*10^{16}\,\mathrm{Sinh}\left[1.5191*10^{-16}\left(8.54375*10^{15}-4.08404*10^{14}x\right)\right]\right)$$

$$y = \int_0^s \sin[\arctan[(61.4-34.09187727106603`-\sigma)/(-16.11843349193664`)]]\, d\sigma$$



The final solution and its graph versus a picture of the actual system.

Part II: The Hanging Spring

The energy in the hanging spring is described by three separate integrals over the same limits. The integrals describe bending energy (which is important in this system, unlike the chain system), spring potential energy, and gravitational potential energy.

$$E = g\int_0^b \rho[x]y[x]\,dx + \int_0^b \Phi[K]\,dx + k\int_0^b s\,dx$$

The energy equation. Φ(k) is a function of curvature that will be added in shortly. The density, as it appears here, is still variable.

This system was parameterized with respect to the arc length of the wire so that the variable density that would be necessary in an equation along the core curve could be incorporated into the function that has a variation applied to it. Having two unknown functions with one dependent on the other would make the problem unsolvable with our current exposure to calculus of variations.

$$\gamma = (x[t], y[t])$$

The parameterization that was used.

The next step was to modify this energy with the above parameterization.

The gravitational potential energy, once parameterized by the arc length of the wire, no longer depended on a variable density because the parameterization traced the length of the wire rather than the core curve. The new function is dependent on a constant density for the whole spring. This value was measured in the lab.

$$G[x[t], y[t]] = \frac{g\rho y[t]}{|\gamma'|} = \frac{g\rho y[t]}{\sqrt{x'[t]^2 + y'[t]^2}}$$

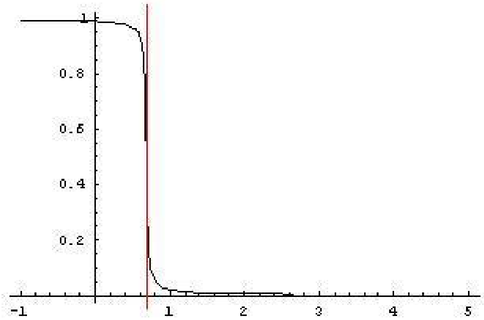
The new gravity function. ρ is the density of the whole spring.

Bending energy is equal to the square of curvature, but in our system, the spring bends such that it can reach a critical curvature, 1/r (r is the internal radius of the spring), where the spring is bent to a maximum. To describe this critical value, it was necessary to impose a function upon our energy such that the energy was very high at these critical values. This was done by dividing the bending energy function by another function of curvature, namely, an arctan function.

$$r = 1.45$$

$$1/r = 0.689655$$

$$F[K] = 0.5\text{`} + \frac{\arctan[-16\pi(-0.6896551724137931\text{`} + K)]}{\pi}$$



The function imposed on the bending energy and its graft. The function, for the most part was determined graphically.

$$K = \frac{-y'[t]\,x''[t] + x'[t]\,y''[t]}{\sqrt{(x'[t]^2 + y'[t]^2)^3}}$$

The formula for curvature in parametric form.

$$B[x[t], y[t]] = \frac{(K)^2}{F[K]} = \frac{(K)^2}{\left(0.5\text{`} - \frac{\text{ArcTan}[16\pi(K - 0.6896551724137931\text{`})]}{\pi}\right)}$$

$$= \frac{(-y'[t]\,x''[t] + x'[t]\,y''[t])^2}{\left(0.5\text{`} - \frac{\text{ArcTan}\left[16\pi\left(-0.6896551724137931\text{`} + \frac{-y'[t]x''[t]+x'[t]y''[t]}{\sqrt{(x'[t]^2+y'[t]^2)^3}}\right)\right]}{\pi}\right)(x'[t]^2 + y'[t]^2)^3}$$

The modified bending energy.

The final energy, spring energy, is a function of arc length. The spring constant was determined in the lab using a summation of forces method.

$$S[x[t], y[t]] = k|\gamma'| = k\sqrt{x'[t]^2 + y'[t]^2}$$

The spring energy.

The energy equation is now complete and the first variation can be applied. It is necessary to apply a separate variation to both the horizontal and vertical parameters because they are separate functions. This method will yield two Euler-Lagrange equations.

Gravitational Potential Energy:

$$G1[x[t],\,y[t]] = \frac{d}{d\xi}G[x[t] + \xi\,\eta[t],\;y[t]]\Big|_0 = \frac{d}{d\xi}\Big[\frac{g\rho\,y[t]}{\sqrt{y'[t]^2 + (x'[t] + \xi\,\eta'[t])^2}}\Big]\Big|_0 = -\frac{g\rho\,y[t]\,x'[t]\,\eta'[t]}{(x'[t]^2 + y'[t]^2)^{3/2}}$$

$$G2[x[t],\,y[t]] = \frac{d}{d\xi}G[x[t],\;y[t] + \xi\,\eta[t]]\Big|_0 = \frac{d}{d\xi}\Big[\frac{g\rho\,(y[t] + \xi\,\eta[t])}{\sqrt{x'[t]^2 + (y'[t] + \xi\,\eta'[t])^2}}\Big]\Big|_0$$

$$= \frac{g\rho\,\eta[t]}{\sqrt{x'[t]^2 + y'[t]^2}} - \frac{g\rho\,y[t]\,y'[t]\,\eta'}{(x'[t]^2 + y'[t]^2)}$$

The two gravitation energies with variations applied.

Bending Energy:

$$B1[x[t],\,y[t]] = \frac{d}{d\xi}B[x[t] + \xi\eta[t],\;y[t]]\Big|_0$$

$$= \frac{d}{d\xi}\Bigg[\frac{((x'[t] + \xi\,\eta'[t])\,y''[t] - y'[t]\,(x''[t] + \xi\,\eta''[t]))^2}{\left(0.5` - \frac{\mathrm{ArcTan}\left[16\,\pi\left(-0.6896 + \frac{(x'[t]+\xi\,\eta'[t])\,y''[t] - y'[t]\,(x''[t]+\xi\,\eta''[t])}{\sqrt{(y'[t]^2+(x'[t]+\xi\,\eta'[t])^2)^3}}\right)\right]}{\pi}\right)(y'[t]^2 + (x'[t] + \xi\,\eta'[t])^2)^3}\Bigg]\Bigg|_0$$

$$= -\frac{6\,x'[t]\,\eta'[t]\,(-y'[t]\,x''[t] + x'[t]\,y''[t])^2}{\left(0.5` - \frac{\mathrm{ArcTan}\left[16\,\pi\left(-0.6896 + \frac{-y'[t]\,x''[t]+x'[t]\,y''[t]}{\sqrt{(x'[t]^2+y'[t]^2)^3}}\right)\right]}{\pi}\right)(x'[t]^2 + y'[t]^2)^4} +$$

$$\frac{2\,(-y'[t]\,x''[t] + x'[t]\,y''[t])\,(\eta'[t]\,y''[t] - y'[t]\,\eta''[t])}{\left(0.5` - \frac{\mathrm{ArcTan}\left[16\,\pi\left(-0.6896 + \frac{-y'[t]\,x''[t]+x'[t]\,y''[t]}{\sqrt{(x'[t]^2+y'[t]^2)^3}}\right)\right]}{\pi}\right)(x'[t]^2 + y'[t]^2)^3} +$$

$$\frac{16\,(-y'[t]\,x''[t] + x'[t]\,y''[t])^2\left(-\frac{3\,x'[t]\,(x'[t]^2+y'[t]^2)^2\,\eta'[t]\,(-y'[t]\,x''[t]+x'[t]\,y''[t])}{((x'[t]^2+y'[t]^2)^3)^{3/2}} + \frac{\eta'[t]\,y''[t]-y'[t]\,\eta''[t]}{\sqrt{(x'[t]^2+y'[t]^2)^3}}\right)}{\left(0.5` - \frac{\mathrm{ArcTan}\left[16\,\pi\left(-0.6896 + \frac{-y'[t]\,x''[t]+x'[t]\,y''[t]}{\sqrt{(x'[t]^2+y'[t]^2)^3}}\right)\right]}{\pi}\right)^2 (x'[t]^2 + y'[t]^2)^3\left(1 + 256\,\pi^2\left(-0.6896 + \frac{-y'[t]\,x''[t]+x'[t]\,y''[t]}{\sqrt{(x'[t]^2+y'[t]^2)^3}}\right)^2\right)}$$

$$B2[x[t],\ y[t]] = \frac{d}{d\xi} B[x[t],\ y[t] + \xi\eta[t]]\Big|_0$$

$$= \frac{d}{d\xi}\left[\frac{(-(y'[t]+\xi\,\eta'[t])\,x''[t]+x'[t]\,(y''[t]+\xi\,\eta''[t]))^2}{\left(0.5\text{`} - \frac{\mathrm{ArcTan}\left[16\pi\left(-0.6896+\frac{-(y'[t]+\xi\,\eta'[t])\,x''[t]+x'[t]\,(y''[t]+\xi\,\eta''[t])}{\sqrt{(x'[t]^2+(y'[t]+\xi\,\eta'[t])^2)^3}}\right)\right]}{\pi}\right)(x'[t]^2+(y'[t]+\xi\,\eta'[t])^2)^3}\right]\Bigg|_0$$

$$= -\frac{6\,y'[t]\,\eta'[t]\,(-y'[t]\,x''[t]+x'[t]\,y''[t])^2}{\left(0.5\text{`} - \frac{\mathrm{ArcTan}\left[16\pi\left(-0.6896+\frac{-y'[t]\,x''[t]+x'[t]\,y''[t]}{\sqrt{(x'[t]^2+y'[t]^2)^3}}\right)\right]}{\pi}\right)(x'[t]^2+y'[t]^2)^4} +$$

$$\frac{2\,(-y'[t]\,x''[t]+x'[t]\,y''[t])\,(-\eta'[t]\,x''[t]+x'[t]\,\eta''[t])}{\left(0.5\text{`} - \frac{\mathrm{ArcTan}\left[16\pi\left(-0.6896+\frac{-y'[t]\,x''[t]+x'[t]\,y''[t]}{\sqrt{(x'[t]^2+y'[t]^2)^3}}\right)\right]}{\pi}\right)(x'[t]^2+y'[t]^2)^3} +$$

$$\frac{16\,(-y'[t]\,x''[t]+x'[t]\,y''[t])^2\left(-\frac{3\,y'[t]\,(x'[t]^2+y'[t]^2)^2\,\eta'[t]\,(-y'[t]\,x''[t]+x'[t]\,y''[t])}{((x'[t]^2+y'[t]^2)^3)^{3/2}}+\frac{-\eta'[t]\,x''[t]+x'[t]\,\eta''[t]}{\sqrt{(x'[t]^2+y'[t]^2)^3}}\right)}{\left(0.5\text{`} - \frac{\mathrm{ArcTan}\left[16\pi\left(-0.6896+\frac{-y'[t]\,x''[t]+x'[t]\,y''[t]}{\sqrt{(x'[t]^2+y'[t]^2)^3}}\right)\right]}{\pi}\right)^2(x'[t]^2+y'[t]^2)^3\left(1+256\,\pi^2\left(-0.6896+\frac{-y'[t]\,x''[t]+x'[t]\,y''[t]}{\sqrt{(x'[t]^2+y'[t]^2)^3}}\right)^2\right)}$$

The two bending energies with variations applied.

Spring Potential Energy:

$$S1[x[t],\ y[t]] = \frac{d}{d\xi} S[x[t] + \xi\eta[t],\ y[t]]\Big|_0 = \frac{d}{d\xi}\left[k\sqrt{y'[t]^2 + (x'[t] + \xi\,\eta'[t])^2}\,\right]\Big|_0 = \frac{k\,x'[t]\,\eta'[t]}{\sqrt{x'[t]^2 + y'[t]^2}}$$

$$S2[x[t],\ y[t]] = \frac{d}{d\xi} S[x[t],\ y[t] + \xi\eta[t]]\Big|_0 = \frac{d}{d\xi}\left[k\sqrt{x'[t]^2 + (y'[t] + \xi\,\eta'[t])^2}\,\right]\Big|_0 = \frac{k\,y'[t]\,\eta'[t]}{\sqrt{x'[t]^2 + y'[t]^2}}$$

The two spring energies with variations applied.

There are also two constraints on this system, each based on the maximum value of the parameter (e.g. the final condition or the second point).

$$C1[x[t], y[t]] = \frac{d}{d\xi} C[x[t] + \xi\eta[t], y[t]]\Big|_0 = \frac{d}{d\xi} [\mu y'[t] + \lambda (x'[t] + \xi \eta'[t])]\Big|_0 = \lambda \eta'[t]$$

$$C2[x[t], y[t]] = \frac{d}{d\xi} C[x[t], y[t] + \xi\eta[t]]\Big|_0 = \frac{d}{d\xi} [\lambda x'[t] + \mu (y'[t] + \xi \eta'[t])]\Big|_0 = \mu \eta'[t]$$

The constraints on the system with variations applied.

The final energy equations look like the following:

$$E1[x[t], y[t]] = \int_0^L (G1[x[t], y[t]] + B1[x[t], y[t]] + S1[x[t], y[t]] + C1[x[t], y[t]])\, dt$$

$$E2[x[t], y[t]] = \int_0^L (G2[x[t], y[t]] + B2[x[t], y[t]] + S2[x[t], y[t]] + C2[x[t], y[t]])\, dt$$

Physical Attributes of the Spring Studied:

Radius of Coil = r = 1.45 cm
Number of Coils = n = 103
Length of Wire = l = $2\pi rn$ = 938.393726 cm
Mass of Spring = m = 21.6578 g
Spring Constant = κ = mg/x = 38.6188*g/(147.32 cm) = 6.25571614 N/cm
Density of Wire = ρ = m/l = 21.6578 g/938.393726 cm = .0230796513 g/cm

Integration by parts was then done by hand to take the etas out of each of these energies. By the fundamental lemma of calculus of variations, the etas were then removed and the resulting equations were then set equal to zero. These equations are the Euler-Lagrange equations for the system. Because the Euler-Lagrange equations are more than a page in length each, they have not been included in the paper and can be viewed on the web.

This is as far as we have taken our work on solving the hanging spring functional.

Conclusions:

The Hanging Chain System:

Solving the hanging chain system parameterized with respect to arc length yielded little error. The hanging systems and the calculated curves have nearly identical minimums and many other points along the calculated curve are consistent with the measured curve from the actual hanging system. What error is present is likely a result of the bending energy associated with the interaction between the links.

The Hanging Spring System:

Because the hanging spring is parameterized with respect to the curve of the wire, it was not possible to simplify the expression by further parameterizing with respect to angle of inclination. This resulted in two problems. In the hanging chain system, the change of variables that allowed for the use of inclination angle, as the function to be varied, reduced the order of the Euler-Lagrange equation by two (x' was replaced by cosθ and y' was replaced by sinθ such that, when integration by parts was applied to the system, two derivatives of x and y were avoided). This left us with a trigonometric equation to solve for our solution, but, because we were unable to further parameterize by angle of inclination in the spring system and because the curvature function required second derivatives, we were left with fourth order Euler-Lagrange equations. The second problem was that, since the energy was now based on two functions, two variations were necessary. The final result of applying calculus of variations to solve the hanging spring functional would be two fourth order differential equations.

We were unable to solve this system. We believe that this system can be solved using numerical methods, in particular, the NDSolve function that is a part of Mathematica. That function requires a number of initial conditions we were unable to find. We believe these values can be obtained through a search where combinations of values are tested to see if they satisfy certain properties of the system (e.g. the "shooting method'). This method could also be simplified with a better knowledge of mechanics. For example we were unable to determine what relationship existed between the second and third derivatives of the arc length of the wire and the physical system.

The largest problem we faced in trying to solve the hanging spring system was a lack of understanding of partial differential equations. The Euler-Lagrange equations that we had to deal with were extremely complex partial differential equations. Broadening our knowledge in this subject along with studying numerical methods would likely make this problem within our ability to solve.